resentation, and x_1,x_2 when we use it as a lower representation, we have contravariant and covariant components of a vector. The representation of a tensor appears as a multilinear form the variables of which are these components of vectors; we get different coefficients for a form representing a given tensor depending on whether we use co- or contravariant components for the vectors; these coefficients are contra- or covariant or mixed components of tensors. The absolute differentiation arises out of the question: given a representation of a tensorfield to find the representation of its differential. The g_{ij} are a representation of the simple tensor $\epsilon(A, x_A, y_A) = x_A.y_A$ and the relations (12) account for all the rules for passing from covariant to contravariant components, etc. The tensor ϵ itself is a most innocent thing: it has the same properties in all points—it is the same tensor indeed in all points, in accordance with which its differential is zero. The important part played by its representation g_{ij} in the usual theory is based on the fact that the properties of the surface are involved in the method of representation.

The question arises now whether there can be devised other representations which do not introduce these complications.

GEOMETRIC ASPECTS OF THE ABELIAN MODULAR FUNCTIONS OF GENUS FOUR (III)[11]

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12. *The form* $\binom{112}{112}$.—This form, written symbolically as $(\alpha x)^2(b\tau)(\beta t)$ where t, τ are digredient binary variables and x is a point in S_2, has 2.2.6–1–2.3–8 = 9 absolute projective constants. The locus of points x for which the bilinear form in t, τ factors is the quartic curve C^4, $(\alpha x)^2(\alpha' x)^2(bb')$-$(\beta\beta') = 0$. For fixed τ and variable t the pencil of conics determined by the form has four base-points on C^4 which form a quadruple p_τ; similarly for fixed t and variable τ there is defined on C^4 a quadruple q_t. The quadruples p_τ form a linear series g_4^1 on C^4; the quadruples q_t a series h_1^4; and these series are residual with respect to each other. Conversely a quadruple on a general quartic C^4 determines its linear series, its residual series of quadruples, and thereby projectively determines the original form. If for given t, τ the conic $(\alpha x)^2(b\tau)(\beta t) = 0$ has a double point x then, according to Wirtinger,[7] the locus of such points x, the vertices of the diagonal triangles of either system of co-residual quadruples, is a plane sextic of genus 4 whose special canonical series are determined by $(\alpha\alpha'\alpha'')^2$ $(b\tau)$-

$(b'\tau)(b''\tau)(\beta t)(\beta' t)(\beta'' t) = 0$. Wirtinger obtains the equation of this sextic as the discriminant of the conic $(\alpha x)(\alpha x_1)(\alpha' x)(\alpha' x_1)(bb')(\beta\beta') = 0$ in variables x_1.

13. *The form* $\binom{112}{112}$ *as the equivalent of a Steiner surface and a quadric.*—Proceeding from the results above we observe that the conics defined by the form $\binom{112}{112}$ lie in a *web* which can be exhibited more symmetrically by replacing the bilinear combinations of t,τ by the coördinates of a point y in S_3. The form then becomes $(\alpha x)^2(ay) = 0$. To a point x of S_2 there corresponds a plane in S_3 which, as x runs over its S_2, envelops a Steiner quartic envelope whose reciprocal point locus is a Steiner four-nodal cubic surface. The effect of replacing the coördinates y by t,τ is to introduce into the S_3 a quadric I with generators t,τ. Then I cuts the Steiner cubic surface in a point sextic of genus 4. Moreover I has in common with the Steiner quartic envelope an octavic locus of planes of genus 3 and the tetrads of planes of this locus which contain the generators t,τ mark on the locus of genus 3 the two residual series of coresidual quadruples. From these considerations one may show that the projective peculiarity of the birationally general Wirtinger sextic, W, of genus 4 is that *its six nodes are the vertices of a four line.* The pencil of conics on this four line is the pencil apolar to the original web. When the plane is mapped by cubic curves on the six vertices it becomes a four-nodal cubic surface and the W-sextic maps into a general space sextic of genus 4 on a quadric I, while the quartic curve C^4 maps into a curve of order 12 on the Steiner cubic surface along which the tangent planes of the surface form the complete octavic intersection of the Steiner envelope and I. The quadric $(\alpha\alpha' u)(\alpha\alpha' v)(ay)(a'y) = 0$ meets the space sextic in the 12 points which are the maps of the points in which the lines u, v of the plane meet W whence $(\alpha\alpha' u)^2(ay)(a'y) = 0$ is, for variable u, a system of contact quadrics of the space sextic. Since each of these is associated with one of the 255 proper half periods of the allied theta functions of genus 4 we have the beautiful theorem:—*On the general space sextic of genus 4 there are precisely 255 four-nodal cubic surfaces each associated with one of the half periods of the corresponding theta functions.*

If u is one of the 28 double tangents of C^4 the contact quadric $(\alpha\alpha' u)^2(ay)(a'y) = 0$ is a contact cone of the space sextic which with the quadric I lies in a pencil of which one member is a pair of tritangent planes. The 28 pencils of this sort give rise to the so-called Steiner complex[12] of 28 pairs of tritangent planes. With respect to each of these 255 Steiner complexes the following theorems may be proved. (*a*) The 28 pairs of tritangent planes of a space sextic of genus 4 which form a Steiner complex are pairs of a Cremona involution in space with the tropes of the Steiner quartic envelope as F-planes. The 28 lines in which the pairs of planes meet are lines of a cubic complex.

The analogous theorem for the double tangents of a plane quartic is that the six pairs of double tangents of a Steiner complex meet in six points on a conic.

The nodal tetrahedron of the Steiner cubic surface has edges on the surface and therefore bisecant to the space sextic. Given a space sextic and such a bisecant tetrahedron the cubic surface can be reconstructed whence (*b*) For a space sextic of genus 4 there are 255 tetrahedra whose edges are bisecants of the curve. The 28 pairs of planes of a Steiner complex cut an edge of the nodal tetrahedron of the allied Steiner cubic in pairs of points of an involution determined by the pair of nodes on that edge and by the pair of crossings of the sextic on that edge.

The Wirtinger scheme in the plane is determined by the general sextic surface with four-fold points and a plane section. For the cubic Cremona involution with F-points at the four nodes of a Steiner cubic transforms the Steiner cubic into a plane and the quadric I into such a sextic surface.

If C_1, C_2 are two Steiner cubic surfaces on the space sextic then $C_1 + kC_2 = \pi I$. Hence C_1 and C_2 cut the plane π in the same cubic curve C_{12}. Moreover on C_{12} in π the nodal tetrahedra of C_1 and C_2 cut out two inscribed four lines of C_{12}. Two cases are possible according as these two four lines in C_{12} correspond to the same or to different half periods on the elliptic cubic C_{12}. Either of these cases can occur. For given the plane, the cubic curve, and two inscribed four lines belonging to the same or to different systems, two Steiner cubic surfaces can be found, each in ∞^4 ways, with this same plane section and given four lines. These two surfaces meet further in a space sextic of genus 4.

Two half periods of the space sextic associated with the surfaces C_1, C_2 determine a third associated with a surface C_3 in either the syzygetic or the azygetic way. Two cases are possible.

Either $\begin{aligned} C_2 + k_{23}C_3 &= \pi_{23}I, \\ C_3 + k_{31}C_1 &= \pi_{31}I, \\ C_1 + k_{12}C_2 &= \pi_{12}I; \end{aligned}$ or $\begin{aligned} k_2C_2 + k_3C_3 &= \pi I, \\ l_3C_3 + l_1C_1 &= \pi I, \\ m_1C_1 + m_2C_2 &= \pi I. \end{aligned}$

In the first case the three surfaces are not in a pencil; in the second case they are. If the first case occurs the two inscribed four lines must belong to the same half period of C_{23} on π_{23}. Otherwise on C_{23} there would be two distinct half periods isolated but not the third—a lack of symmetry not to be expected. Presumably then the second case occurs when on the plane π the nodal tetrahedra of C_1, C_2, C_3 meet π in inscribed four lines, one from each of the three systems. With respect to the first case we may prove the theorem: (*c*) Given two tetrahedra in general position T_1, T_2, a plane τ can be chosen in four ways such that the edges of T_1, T_2 meet π in the two sets of 6 vertices of two four lines inscribed in a cubic curve

C_{12} and belonging to the same half period on $\dot{C}_{12}$. There are two Steiner cubic surfaces C_1, C_2 with nodal tetrahedra T_1, T_2 respectively and the plane section C_{12} such that $k_1C_1 + k_2C_2 = \pi I$. The plane π is the plane of one of the four conics in the pencil of quadric envelopes which touch the 8 planes of T_1 and T_2.

14. *Particular cases of W*.—There are four particular cases of interest in connection with the canonical sextic W each characterized by one condition. When one half period or discriminant factor of the sextic is isolated the others are of two kinds, respectively syzygetic or azygetic with the given one. Thus when the space sextic is determined by a Steiner cubic and the quadric I it acquires a node in one of two ways—either I touches the Steiner surface or I passes through a node of the Steiner surface.

In the case where I touches the Steiner surface at the ordinary point p the space sextic has a node at p. The octavic envelope in which I and the Steiner quartic envelope meet also has the tangent plane at p for a double plane with contact at p whence the ternary quartic C^4 has a node at the same point the W-sextic has an extra node.

In the case where I passes through a node of the Steiner cubic the corresponding line in the plane factors out and the W-sextic is a residual quintic of genus 3 with simple points at the three vertices on this line and nodes at the other three vertices. The curve C^4 however has no singularity.

A third particular case which from a certain point of view is not a special case of the W-sextic occurs when I is a quadric cone and the sets t,τ of generators coincide. Here the web of conics in the plane no longer consists of residual pencils but rather contains an isolated conic and a linearly independent quadratic system of ∞^1 conics. If t is a parameter on the isolated conic the quadratic system cuts this conic in a quadratic system of quadruples determined by a general form $(\alpha t)^4(a\tau)^2 = 0$ with only 8 absolute constants and the locus of the vertices of the diagonal triangles of these quadruples is the W-sextic. The curve C^4 is of hyperelliptic type—the isolated conic doubly covered—with 8 branch points, $(\alpha t)^4$-$(\alpha' t)^4(aa')^2 = 0$. A birationally equivalent form of this sextic is the locus of the 9-th node of sextics with 8 given nodes—a 9-ic curve with triple points at the 8 given nodes. When the 8 points are known this 9-ic can be mapped upon the space sextic in such a way that all of the 120 tritangent planes are rationally known.[13] This case is characterized by the vanishing of an even theta function for the zero argument.

The fourth case—defined by the vanishing of the invariant A of section 11—occurs when the two space cubic curves of section 2 are rational point and line cubic in the same plane. For the space sextic of genus 4 two generators t of I meet the sextic at points crossed by the same three generators τ. For the W-sextic the C^4 is generated by two projective pencils of conics on coresidual quadruples in such a way that in the projectivity the three degenerate members of the two pencils correspond.

[11] Abstracts I and II, these PROCEEDINGS, **7**, 1921 (245, 334).
[12] Coble, *Trans. Amer. Math. Soc.*, **14**, 1913 (261).
[13] Coble, *Trans. Amer. Math. Soc.*, **17**, 1916 (358).

ON IRREGULARITIES IN THE VELOCITY CURVES OF SPECTROSCOPIC BINARIES

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The spectrographic velocity curves of several Cepheid variable stars, and of a few other stars not yet definitely known to be variable, show puzzling deviations from simple elliptical orbital motion. These deviations have hitherto been explained by superposing upon a primary elliptical velocity curve a secondary oscillation (generally circular) whose period must be *precisely* one-half or one-third the period of the primary. Such treatment has produced fairly good agreement with the observed velocity data. There is, however, a similarity in the location of the nodes of the primary and secondary curves which savors of artificiality. Moreover, it is very difficult, if not impossible, to devise reasonable and stable multiple systems, or tidally distorted stars, which shall produce such anomalies of one-half or one-third the period of the system.

My failure to construct a dynamically reasonable tidal or other model giving such oscillations in an exact submultiple of the period has led me to the attempt to represent the observational data by means of a purely elliptical velocity curve, plus a *single* oscillation or "hump." This new method of treatment is of considerable interest and, if substantiated by future more accurate spectrographic data, will have an important bearing upon some of the many theories of Cepheid variation, the most puzzling feature of which, as is well known, is the essential synchronism of maximum light with maximum velocity of approach.

Limitations of space make it impossible to give in this paper the curves which have been derived for these stars under the hypothesis of secondaries of a submultiple of the period; these curves will be found in the original papers, references to which are given at the close. In cuts I–VI, shown in Figures 1 and 2, the original spectrographic velocities are plotted without change as given by the respective observers, the dots representing three-prism results, and the open circles those derived from one-prism